1.3b First order linear difference equations

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Consider a first-order linear difference eqn

$$x_{t+1} = a_t x_t + b_t$$

Given initial condition x_0, this has a unique solution

x_0

$x_1 = a_0 x_0 + b_0$

$x_2 = a_1 x_1 + b_1 = a_1(a_0 x_0 + b_0) + b_1 = a_1 a_0 x_0 + a_1 b_0 + b_1$

$x_3 = a_2 x_2 + b_2 = a_2(a_1 a_0 x_0 + a_1 b_0 + b_1) + b_2 = a_2 a_1 a_0 x_0 + a_2 a_1 b_0 + a_2 b_1 + b_2$

$\vdots$

$$x_{t+1} = \left(\prod_{i=0}^{t} a_i\right) x_0 + b_t + \sum_{i=0}^{t-1} b_i \prod_{j=i+1}^{t} a_j$$

Suppose $a_t \equiv a$ and $b_t \equiv b$ constants. Then

$$x_{t+1} = a x_t + b$$

$t+1$

$$x_{t+1} = a^{t+1} x_0 + b \sum_{i=0}^{t} a^i$$

$$\Rightarrow \quad x_{t+1} = \begin{cases} x_0 a^{t+1} + b \cdot \dfrac{1-a^{t+1}}{1-a}, & a \neq 1 \\ x_0 + (t+1) b, & a = 1. \end{cases}$$

Principle of Superposition

Suppose x_{t} satisfies $x_{t+1} = a x_t$ $(b=0)$

Principle of superposition

Suppose $\underbrace{x_{h,t}}_{\text{homogeneous sol}}$ satisfies $x_{h,t+1} = a x_{h,t}$ $\quad (b=0)$

Suppose $\underbrace{x_{p,t}}_{\text{particular}}$ satisfies $x_{p,t+1} = a x_{p,t} + b$ $\leftarrow b \neq 0$

Then $x_t = x_{h,t} + x_{p,t}$ satisfies $x_{t+1} = a x_t + b$ by linearity.

proof. $$x_{t+1} = x_{h,t+1} + x_{p,t+1} = a x_{h,t} + a x_{p,t} + b$$
$$= a\left(x_{h,t} + x_{p,t}\right) + b$$
$$= a x_t + b. \quad \square$$

Method of undetermined coefficients

Note $x_{h,t} = ca^t$ solves $x_{h,t+1} = a x_{h,t}$ $\forall c \in \mathbb{R}$.

Need to find x_p. Guess $x_p = k$ for some constant k. Here, b is constant

If $a \neq 1$, this works, and $k = ak + b \implies k = \frac{b}{1-a}$

Guessing $x_p = k \implies k = ak + b \implies k = \frac{b}{1-a}$

If $a = 1$, guess $x_p = tk$. Then $(t+1)k = tk + b$
$$\implies k = b$$

Suppose $x_p = k$. $k = ak + b$
$\implies 0 \neq b$. $x_p = k$ is a sol. to the homogeneous eqn

So $$x_t = \begin{cases} ca^t + \dfrac{b}{1-a}, & a \neq 1 \\ c + tb, & a = 1 \end{cases}$$ solves $x_{t+1} = a x_t + b$

(general solution)

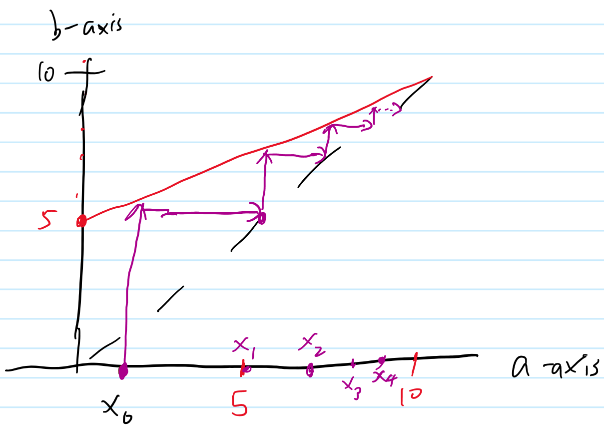
Cobwebbing

Suppose $x_{t+1} = f(x_t)$, a first-order equation

Then $y = f(x)$ is the reproduction curve.

Perform the following algorithm:

- Graph the lines $y = x$ and $y = f(x)$
- Let $(a, b) = \underbrace{(x_0, 0)}$ be the starting point
- Repeat the following steps while moving to the new (a, b)
  - $b \leftarrow f(a)$
  - $a \leftarrow b$



$$f(x) = \frac{1}{2}x + 5$$